Trigonometria II

Transformações de Soma em Produto

2° ano E.M.

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1 Exercícios Introdutórios

Exercício 1. Sejam p e q dois ângulos quaisquer, podemos afirmar que $\cos p + \cos q$ é igual a:

a) $2 \cdot \cos \dfrac{p+q}{2} \cdot \cos \dfrac{p-q}{2}$.

b) $-2 \cdot \operatorname{sen} \dfrac{p+q}{2} \cdot \operatorname{sen} \dfrac{p-q}{2}$.

c) $2 \cdot \operatorname{sen} \dfrac{p+q}{2} \cdot \cos \dfrac{p-q}{2}$.

d) $2 \cdot \cos \dfrac{p+q}{2} \cdot \operatorname{sen} \dfrac{p-q}{2}$.

Exercício 2. $2 \cdot \operatorname{sen} \dfrac{p+q}{2} \cdot \cos \dfrac{p-q}{2}$ é igual a:

a) $\cos p + \cos q$.

b) $\operatorname{sen} p + \operatorname{sen} q$.

c) $\cos p - \cos q$.

d) $\operatorname{sen} p - \operatorname{sen} q$.

Exercício 3. $\operatorname{sen}(3a) - \operatorname{sen} a$ é igual a:

a) $2\cos(2a) \cdot \operatorname{sen} a$.

b) $2\operatorname{sen}(2a) \cdot \operatorname{sen} a$.

c) $2\cos(2a) \cdot \cos a$.

d) $-2\operatorname{sen}(2a) \cdot \operatorname{sen} a$.

e) $-2\cos(2a) \cdot \operatorname{sen} a$.

Exercício 4. A expressão $\dfrac{\operatorname{sen} x + \operatorname{sen} y}{\operatorname{sen} x - \operatorname{sen} y}$ é equivalente a:

a) $\operatorname{sen} \dfrac{x+y}{2} \cdot \operatorname{cotg} \dfrac{x-y}{2}$.

b) $\operatorname{cotg} \dfrac{x+y}{2} \cdot \operatorname{tg} \dfrac{x-y}{2}$.

c) $\operatorname{tg} \dfrac{x+y}{2} \cdot \operatorname{cotg} \dfrac{x-y}{2}$.

d) $\operatorname{tg} \dfrac{x+y}{2} \cdot \operatorname{tg} \dfrac{x-y}{2}$.

e) $\cos \dfrac{x+y}{2} \cdot \operatorname{sen} \dfrac{x-y}{2}$.

Exercício 5. Se $\operatorname{sen} p + \operatorname{sen} q = 2 \cdot \operatorname{sen} a \cdot \cos b$, então $a + b$ é igual a:

a) p.

b) q.

c) $p + q$.

d) $p - q$.

e) $2p + 2q$.

2 Exercícios de Fixação

Exercício 6. Mostre que $\operatorname{sen} 20^o + \operatorname{sen} 40^o = \operatorname{sen} 80^o$.

Exercício 7. Transforme o produto $\operatorname{sen} 75^o \cdot \cos 15^o$ em soma e resolva-o.

Exercício 8. Mostre que $\dfrac{\cos x}{\cos 4x + \cos 2x} = \dfrac{\sec 3x}{2}$.

Exercício 9. Determine os valores de $x \in [0, \pi]$, para os quais $\operatorname{sen} 3x + \operatorname{sen} x = 0$.

Exercício 10. Transforme o produto $\operatorname{sen} \dfrac{13\pi}{12} \cdot \cos \dfrac{11\pi}{12}$ em soma e calcule seu valor.

Exercício 11. O número de valores de x, tal que $x \in [0, 2\pi]$, para os quais $\operatorname{sen} 3x = \operatorname{sen} 7x$ é:

a) 3.

b) 8.

c) 13.

d) 18.

e) 23.

Exercício 12. Mostre que $\dfrac{\operatorname{sen} 30^o + \operatorname{sen} 40^o + \operatorname{sen} 50^o}{\cos 30^o + \cos 40^o + \cos 50^o} = \operatorname{tg} 40^o$.

Exercício 13. Escreva na forma de produto a expressão $\operatorname{sen} \dfrac{3x}{2} - \operatorname{sen} \dfrac{x}{2}$.

3 Exercícios de Aprofundamento e de Exames

Exercício 14. Determine $S = \cos x + \cos 2x + \cos 3x + ... + \cos(nx)$.

Exercício 15. O valor $y = \operatorname{sen} 70^o \cdot \cos 50^o + \operatorname{sen} 260^o \cdot \cos 280^o$ é:

a) $\sqrt{3}$.

b) $\dfrac{\sqrt{3}}{2}$.

c) $\dfrac{\sqrt{3}}{3}$.

d) $\dfrac{\sqrt{3}}{4}$.

e) $\dfrac{\sqrt{3}}{5}$.

Exercício 16. Mostre que $\operatorname{sen}^2(nx) = \dfrac{1 - \cos(2nx)}{2}$.

Exercício 17. A sequência $\operatorname{sen} 15^o$, $\operatorname{sen} a$ e $\operatorname{sen} 75^o$ formam nesta ordem uma PA, encontre $\operatorname{sen} a$.

a) $\dfrac{\sqrt{6}}{2}$.

b) $\dfrac{\sqrt{6}}{4}$.

c) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$.

d) $\dfrac{1}{2}$.

e) $\operatorname{sen} 45^o$.

Exercício 18. Para todo $x \in \mathbb{R}$, a expressão $\cos^2(2x) \cdot \operatorname{sen}^2(2x) \cdot \operatorname{sen} x$ é igual a:

a) $2^{-4}[\operatorname{sen}(2x) + \operatorname{sen}(5x) + \operatorname{sen}(7x)]$.

b) $2^{-4}[2\operatorname{sen} x + \operatorname{sen}(7x) - \operatorname{sen}(9x)]$.

c) $2^{-4}[-\operatorname{sen}(2x) - \operatorname{sen}(3x) + \operatorname{sen}(7x)]$.

d) $2^{-4}[-\operatorname{sen} x + 2\operatorname{sen}(5x) - \operatorname{sen}(9x)]$.

e) $2^{-4}[\operatorname{sen} x + 2\operatorname{sen}(3x) + \operatorname{sen}(5x)]$.

Exercício 19. Calcule $\cos(x-y)$ em função de a e b, sabendo que o produto $ab \neq 0$, que $\operatorname{sen} x + \operatorname{sen} y = a$ e que $\cos x + \cos y = b$.

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Respostas e Soluções.

1. A.

2. B.

3.

$$\begin{aligned}\operatorname{sen}(3a)-\operatorname{sen}a &= 2\cdot\cos\frac{3a+a}{2}\cdot\operatorname{sen}\frac{3a-a}{2}\\ &= 2\cdot\cos(2a)\cdot\operatorname{sen}a.\end{aligned}$$

Resposta A.

4.

$$\begin{aligned}\frac{\operatorname{sen}x+\operatorname{sen}y}{\operatorname{sen}x-\operatorname{sen}y} &= \frac{2\cdot\operatorname{sen}\frac{x+y}{2}\cdot\cos\frac{x-y}{2}}{2\cdot\cos\frac{x+y}{2}\cdot\operatorname{sen}\frac{x-y}{2}}\\ &= \operatorname{tg}\frac{x+y}{2}\cdot\operatorname{cotg}\frac{x-y}{2}.\end{aligned}$$

Resposta C.

5. Como $\operatorname{sen}p+\operatorname{sen}q = 2\cdot\operatorname{sen}\frac{p+q}{2}\cdot\cos\frac{p-q}{2}$, então $a=\frac{p+q}{2}$ e $b=\frac{p-q}{2}$, donde $a+b=p$. Resposta A.

6. (Extraído da Vídeo Aula)

$$\begin{aligned}\operatorname{sen}20^o+\operatorname{sen}40^o &= 2\cdot\operatorname{sen}\frac{20^o+40^o}{2}\cdot\cos\frac{20^o-40^o}{2}\\ &= 2\cdot\operatorname{sen}30^o\cdot\cos(-10^o)\\ &= 2\cdot\operatorname{sen}30^o\cdot\cos10^o\\ &= 2\cdot\frac{1}{2}\cdot\cos10^o\\ &= \cos10^o\\ &= \operatorname{sen}80^o.\end{aligned}$$

7. Seja $\frac{p+q}{2}=75^o$ e $\frac{p-q}{2}=15^o$, então $p=90^o$ e $q=60^o$. Temos assim:

$$\begin{aligned}\operatorname{sen}75^o\cdot\cos15^o &= \frac{2\cdot\operatorname{sen}\frac{90^o+60^o}{2}\cdot\cos\frac{90^o-60^o}{2}}{2}\\ &= \frac{\operatorname{sen}90^o+\operatorname{sen}60^o}{2}\\ &= \frac{1+\frac{\sqrt{3}}{2}}{2}\\ &= \frac{2+\sqrt{3}}{4}.\end{aligned}$$

8.

$$\begin{aligned}\frac{\cos x}{\cos4x+\cos2x} &=\\ \frac{\cos x}{2\cdot\cos\frac{4x+2x}{2}\cdot\cos\frac{4x-2x}{2}} &=\\ \frac{\cos x}{2\cdot\cos3x\cdot\cos x} &=\\ \frac{1}{2\cos3x} &= \frac{\sec3x}{2}.\end{aligned}$$

9.

$$\begin{aligned}\operatorname{sen}3x+\operatorname{sen}x &= 0\\ 2\cdot\operatorname{sen}\frac{3x+x}{2}\cdot\cos\frac{3x-x}{2} &= 0\\ 2\cdot\operatorname{sen}2x\cdot\cos x &= 0\\ \operatorname{sen}2x\cdot\cos x &= 0.\end{aligned}$$

Sendo assim, $\operatorname{sen}2x=0$ ou $\cos x=0$, ou seja, $x_1=0$, $x_2=\frac{\pi}{2}$ e $x_3=\pi$.

10. Sejam a e b tais que $\frac{a+b}{2}=\frac{13\pi}{12}$ e $\frac{a-b}{2}=\frac{11\pi}{12}$, segue que $a=2\pi$ e $b=\frac{\pi}{6}$. Além disso, sabemos que $\operatorname{sen}a+\operatorname{sen}b=2\cdot\operatorname{sen}\frac{a+b}{2}\cdot\cos\frac{a-b}{2}$. Temos, então:

$$\begin{aligned}\operatorname{sen}\frac{13\pi}{12}\cdot\cos\frac{11\pi}{12} &= \frac{2\cdot\operatorname{sen}\frac{13\pi}{12}\cdot\cos\frac{11\pi}{12}}{2}\\ &= \frac{2\cdot\operatorname{sen}\frac{2\pi+\frac{\pi}{6}}{12}\cdot\cos\frac{2\pi-\frac{\pi}{6}}{12}}{2}\\ &= \frac{\operatorname{sen}2\pi+\operatorname{sen}\frac{\pi}{6}}{2}\\ &= \frac{0+\frac{1}{2}}{2}\\ &= \frac{1}{4}.\end{aligned}$$

11.

$$\begin{aligned}\operatorname{sen}3x &= \operatorname{sen}7x\\ \operatorname{sen}7x-\operatorname{sen}3x &= 0\\ 2\cdot\cos\frac{7x+3x}{2}\cdot\operatorname{sen}\frac{7x-3x}{2} &= 0\\ \cos5x\cdot\operatorname{sen}2x &= 0.\end{aligned}$$

Sendo assim, $\cos5x=0$, donde $x=\frac{\pi+2k\pi}{10}$, $k\in\mathbb{Z}$, que implica em 10 soluções; ou $\operatorname{sen}2x=0$, donde $x=\frac{k\pi}{2}$, $k\in\mathbb{Z}$, que implica em 5 soluções, das quais 2 já encontradas na equação anterior. Portanto, no intervalo $[0,2\pi]$, são $10+3=13$ soluções. Resposta C.

12. (Extraído da Vídeo Aula)

$$\begin{aligned}
\frac{\operatorname{sen} 30^o + \operatorname{sen} 40^o + \operatorname{sen} 50^o}{\cos 30^o + \cos 40^o + \cos 50^o} &= \\
\frac{\operatorname{sen} 30^o + \operatorname{sen} 50^o + \operatorname{sen} 40^o}{\cos 30^o + \cos 50^o + \cos 40^o} &= \\
\frac{2 \cdot \operatorname{sen} \dfrac{50^o + 30^o}{2} \cdot \cos \dfrac{50^o - 30^o}{2} + \operatorname{sen} 40^o}{2 \cdot \cos \dfrac{50^o + 30^o}{2} \cdot \cos \dfrac{50^o - 30^o}{2} + \cos 40^o} &= \\
\frac{2 \cdot \operatorname{sen} 40^o \cdot \cos 10^o + \operatorname{sen} 40^o}{2 \cdot \cos 40^o \cdot \cos 10^o + \cos 40^o} &= \\
\frac{\operatorname{sen} 40^o (2 \cos 10^o + 1)}{\cos 40^o (2 \cos 10^o + 1)} &= \\
\frac{\operatorname{sen} 40^o}{\cos 40^o} &= \operatorname{tg} 40^o.
\end{aligned}$$

13.

$$\begin{aligned}
\operatorname{sen} \frac{3x}{2} - \operatorname{sen} \frac{x}{2} &= \\
2 \cdot \cos \frac{\frac{3x}{2} + \frac{x}{2}}{2} \cdot \operatorname{sen} \frac{\frac{3x}{2} - \frac{x}{2}}{2} &= \\
2 \cdot \cos x \cdot \operatorname{sen} \frac{x}{2}.
\end{aligned}$$

14. (Extraído da Vídeo Aula) Temos que:

$$\begin{cases}
\operatorname{sen} \dfrac{3x}{2} - \operatorname{sen} \dfrac{x}{2} & = 2 \operatorname{sen} \dfrac{x}{2} \cos x \\
\operatorname{sen} \dfrac{5x}{2} - \operatorname{sen} \dfrac{3x}{2} & = 2 \operatorname{sen} \dfrac{x}{2} \cos 2x \\
\operatorname{sen} \dfrac{7x}{2} - \operatorname{sen} \dfrac{5x}{2} & = 2 \operatorname{sen} \dfrac{x}{2} \cos 3x \\
\ldots & \\
\operatorname{sen} \dfrac{(2n-1)x}{2} - \operatorname{sen} \dfrac{(2n-3)}{2} & = 2 \operatorname{sen} \dfrac{x}{2} \cos[(n-1)x] \\
\operatorname{sen} \dfrac{(2n+1)x}{2} - \operatorname{sen} \dfrac{(2n-1)}{2} & = 2 \operatorname{sen} \dfrac{x}{2} \cos(nx)
\end{cases}$$

Somando as equações, chegamos a:

$$\begin{aligned}
\operatorname{sen} \frac{(2n+1)x}{2} - \operatorname{sen} \frac{x}{2} &= 2 \operatorname{sen} \frac{x}{2} \cdot S \\
S &= \frac{\operatorname{sen} \dfrac{(2n+1)x}{2} - \operatorname{sen} \dfrac{x}{2}}{2 \operatorname{sen} \dfrac{x}{2}} \\
S &= \frac{2 \cdot \cos \dfrac{(n+1)x}{2} \cdot \operatorname{sen} \dfrac{nx}{2}}{2 \operatorname{sen} \dfrac{x}{2}} \\
S &= \frac{\cos \dfrac{(n+1)x}{2} \cdot \operatorname{sen} \dfrac{nx}{2}}{\operatorname{sen} \dfrac{x}{2}}
\end{aligned}$$

15. (Extraído do IME)

$$\begin{aligned}
y &= \operatorname{sen} 70^o \cdot \cos 50^o + \operatorname{sen} 260^o \cdot \cos 280^o \\
&= \operatorname{sen} 70^o \cdot \cos 50^o - \operatorname{sen} 80^o \cdot \cos 80^o \\
&= \frac{2 \operatorname{sen} 70^o \cdot \cos 50^o - \operatorname{sen} 160^o}{2} \\
&= \frac{\operatorname{sen} 120^o + \operatorname{sen} 20^o - \operatorname{sen} 20^o}{2} \\
&= \frac{\operatorname{sen} 60^o}{2} \\
&= \frac{\sqrt{3}}{4}.
\end{aligned}$$

Resposta D.

16. (Extraído da Vídeo Aula) Sabemos que $\operatorname{sen}^2 \alpha + \cos^2 \alpha = 1$ e $\cos(2\alpha) = \cos^2 \alpha - \operatorname{sen}^2 \alpha$. Temos, então:

$$\begin{aligned}
\cos(2nx) &= (\cos(nx))^2 - (\operatorname{sen}(nx))^2 \\
\cos(2nx) &= 1 - (\operatorname{sen}(nx))^2 - (\operatorname{sen}(nx))^2 \\
\cos(2nx) &= 1 - 2(\operatorname{sen}(nx))^2 \\
2(\operatorname{sen}(nx))^2 &= 1 - \cos(2nx) \\
(\operatorname{sen}(nx))^2 &= \frac{1 - \cos(2nx)}{2}.
\end{aligned}$$

17. (Extraído da VUNESP) Como a sequência forma uma PA, temos:

$$\begin{aligned}
\operatorname{sen} a &= \frac{\operatorname{sen} 15^o + \operatorname{sen} 75^o}{2} \\
&= \frac{2 \cdot \operatorname{sen} \dfrac{75^o + 15^o}{2} \cdot \cos \dfrac{75^o - 15^o}{2}}{2} \\
&= \operatorname{sen} 45^o \cdot \cos 30^o \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{6}}{4}.
\end{aligned}$$

Resposta B.

18. (Extraído do ITA)

$$\begin{aligned}
(\cos(2x))^2 \cdot (\operatorname{sen}(2x))^2 \cdot \operatorname{sen} x &= \\
\left(\frac{1}{2} \cdot \operatorname{sen}(4x)\right)^2 \cdot \operatorname{sen} x &= \\
\frac{1}{4} \cdot \operatorname{sen}(4x) \cdot \operatorname{sen}(4x) \cdot \operatorname{sen} x &= \\
-\frac{1}{8} \cdot (\cos(8x) - \cos 0) \cdot \operatorname{sen} x &= \\
-\frac{1}{8} \cdot (\cos(8x) \cdot \operatorname{sen} x - \cos 0 \cdot \operatorname{sen} x) &= \\
-\frac{1}{16} \cdot (\operatorname{sen}(9x) - \operatorname{sen}(7x) - 2 \operatorname{sen} x) &= \\
\frac{1}{16} \cdot (-\operatorname{sen}(9x) + \operatorname{sen}(7x) + 2 \operatorname{sen} x).
\end{aligned}$$

Resposta B.

19. Como $\operatorname{sen} x + \operatorname{sen} y = a$ e $\cos x + \cos y = b$, temos:

$$
\begin{aligned}
(\operatorname{sen} x + \operatorname{sen} y)^2 + (\cos x + \cos y)^2 &= a^2 + b^2 \\
2(\cos x \cos y + \operatorname{sen} x \operatorname{sen} y) + 2 &= a^2 + b^2 \\
2\cos(x - y) &= a^2 + b^2 - 2 \\
\cos(x - y) &= \frac{a^2 + b^2 - 2}{2}.
\end{aligned}
$$

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